Table 3 Natural frequencies of square isotropic plates with at least two free edges ($\nu = 0.3$)

Boundary conditions	Asymptotic analysis $k_x a$	$k_y a$	$\omega a^2(\rho/D)^{1/2}$	Ref. 7 $\omega a^2(\rho/D)^{1/2}$
Clamped				22.17
($x = 0, x = a$)	1.417	0.838	26.73	26.40
	1.280	1.696	44.56	43.6
Free				61.2
($y = 0, y = a$)	2.465	0.860	67.29	67.2
	1.190	2.598	80.60	79.8
	2.381	1.807	88.17	87.5
Clamped				3.430(L), 3.473(U)[a]
($x = 0$)	0.544	0.702	7.788	7.260(L), 8.547(U)
				20.87(L), 21.30(U)
Free	0.498	1.553	26.27	26.50(L), 27.29(U)
($x = a, y = 0, y = a$)	1.528	0.844	30.06	28.55(L), 31.17(U)
	1.546	1.733	53.21	51.50(L), 54.26(U)
				60.25(L), 61.28(U)
	0.449	2.518	64.54	64.2
	2.513	0.860	69.63	71.1
	1.535	2.641	92.08	
	2.537	1.815	96.06	92.14(L), 97.21(U)
Clamped	0.545	0.545	5.866	6.958(U)
($x = 0, y = 0$)	0.501	1.512	25.05	24.80(U)[b]
	1.512	0.501	25.05	26.80(U)[c]
Free	1.545	1.545	47.13	48.05(U)
($x = a, y = a$)	0.449	2.504	63.87	63.14(U)[b]
	2.504	0.449	63.87	

[a] (L) and (U) denote lower and upper bounds.
[b] Mode antisymmetric with respect to $x = y$.
[c] Mode symmetric with respect to $x = y$.

of the structural symmetry with respect to this diagonal. Young's results[8] reproduced by Leissa[7] and given in Table 3, while not revealing commensurate modes, include two natural frequencies (second and third) which differ by only eight %. The asymptotic analysis similarly suggests that the fifth and sixth natural frequencies are almost equal, but unfortunately the writers have been unable to find values reported for frequencies beyond the fifth.

Although it appears that the lower frequencies are not so accurately predicted as when the transverse displacement vanishes everywhere on the plate boundary, and bearing in mind that beam-like modes are overlooked, we conclude that Bolotin's method is a useful technique for the estimation of natural frequencies of a plate when a portion of the boundary is unsupported.

References

[1] Bolotin, V. V., Makarov, B. P., Mishenkov, G. V., and Shveiko, Yu. Yu., "Asymptotic Method of Investigating the Natural Frequency Spectrum of Elastic Plates," *Raschet na Prochnost*, Mashgiz, Moscow, No. 6, 1960, pp. 231–253 (in Russian).

[2] Bolotin, V. V., "The Edge Effect in the Oscillations of Elastic Shells," *Prikladnaya Matematika i Mekhanika*, Vol. 24, No. 5, 1960, pp. 831–842.

[3] Bolotin, V. V., "Dynamic Edge Effect in the Elastic Vibrations of Plates," *Inshen. Sbornik.*, Vol. 31, 1961, pp. 3–14, (in Russian).

[4] Bolotin, V. V., "An Asymptotic Method for the Study of the Problem of Eigenvalues for Rectangular Regions," *Problems of Continuum Mechanics* (Volume Dedicated to N. I. Muskhelishvili), Society of Industrial and Applied Mathematics, Philadelphia, Pa., 1961, pp. 56–68.

[5] Dickinson, S. M., "The Flexural Vibrations of Rectangular Orthotropic Plates," *Journal of Applied Mechanics*, Vol. 36, No. 1, March 1969, pp. 101–112.

[6] Kanazawa, T. and Kawai, T., "On the Lateral Vibration of Anisotropic Rectangular Plates," *Proceedings of the Second Japanese National Congress of Applied Mechanics*, 1962, pp. 333–338.

[7] Leissa, A. W., *Vibration of Plates*, NASA SP-160, 1969, pp. 72–86.

[8] Young, D., "Vibration of Rectangular Plates by the Ritz Method," *Journal of Applied Mechanics*, Vol. 17, No. 4, Dec. 1950, pp. 448–453.

Calculation of Compressible Turbulent Free Shear Layers

YOUN H. OH*
NASA Langley Research Center, Hampton, Va.

Introduction

NOTWITHSTANDING its well-known shortcomings, Prandtl's mixing length theory has been a valuable engineering tool for the prediction of mean velocity fields in turbulent shear layers. Recently Rudy and Bushnell,[1] employing different values of normalized mixing length for planar and axisymmetric flows, satisfactorily predicted a wide range of free turbulent flows except with large sustained density differences. Considering the success of mixing length theory in compressible wall shear flows,[2,3] Rudy and Bushnell suggested the neglect in their solution procedure of turbulence induced transverse static pressure gradients as a possible cause of the poor predictions in supersonic free shear layers with large density differences. Data reported by Brown and Roshko[4] for binary gas mixing show very small effects of density variation on the spreading of a turbulent mixing layer in low-speed flow. They postulated that the large differences in spreading observed in supersonic free mixing layers is an effect of Mach number rather than an effect of density variation. The motivation of this Note is to resolve some of the questions raised previously and to extend the applicability of Rudy and Bushnell's mixing length theory to include supersonic free turbulent mixing.

In the following analysis the normal momentum equation is coupled with the conventional equations of motion and solved iteratively. Results show the transverse static pressure variation has very little direct effect on the mean flow variables in compressible free turbulent shear layers. Indirect effects of static pressure variation through correlations of the fluctuating pressure and fluctuating velocity field (also mentioned by Rudy and Bushnell as a possible mechanism responsible for the observed decreased mixing at higher Mach number) have to be assessed by higher order closure methods not considered herein. However, recently Bradshaw[5] obtained improved predictions of skin friction for supersonic turbulent boundary layers with pressure gradient by including a "mean dilatation effect" in the shear stress equation. His success prompted the present effort to apply a "mean dilatation" correction factor to the mixing length to see if the Mach number effect on spreading rate of a free shear layer would be correctly accounted for. Results show that the use of a mixing length corrected by a properly weighted average value of Bradshaw's[5] mean dilatation factor predicts supersonic turbulent free shear layers correctly.

Analysis

We consider the turbulent homogeneous mixing of two-dimensional or axisymmetric supersonic jets with a low-speed gas as shown schematically in the insert on Fig. 1. With the usual approximation that streamwise derivatives are small compared with normal derivatives, the conservation equations for the mean quantities may be written in the following form:

$$\partial \bar{\rho}\tilde{u}/\partial x + y^{-j}\partial y^j \bar{\rho}\tilde{v}/\partial y = 0 \tag{1}$$

$$\bar{\rho}\tilde{u}\,\partial\tilde{u}/\partial x + \bar{\rho}\tilde{v}\,\partial\tilde{u}/\partial y + y^{-j}\partial y^j(\overline{\rho u'v'} - R_e{}^{-1}\bar{\mu}\,\partial\tilde{u}/\partial y)/\partial y = 0 \tag{2}$$

$$\partial\bar{p}/\partial y + \partial\overline{\rho v'^2}/\partial y = 0 \tag{3}$$

Received September 6, 1973; revision received November 5, 1973. Work was done at NASA Langley under Contract NAS1-11707 with Old Dominion University. Numerical program was developed during tenure as NRC-NASA Resident Research Associate.

Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions; Supersonic and Hypersonic Flow.

* Research Fellow.

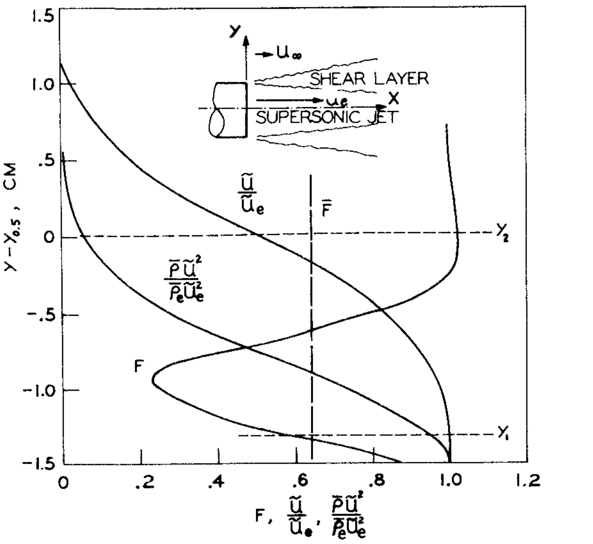

Fig. 1 Compressible mean dilatation correction factor F at $x = 10$ cm.

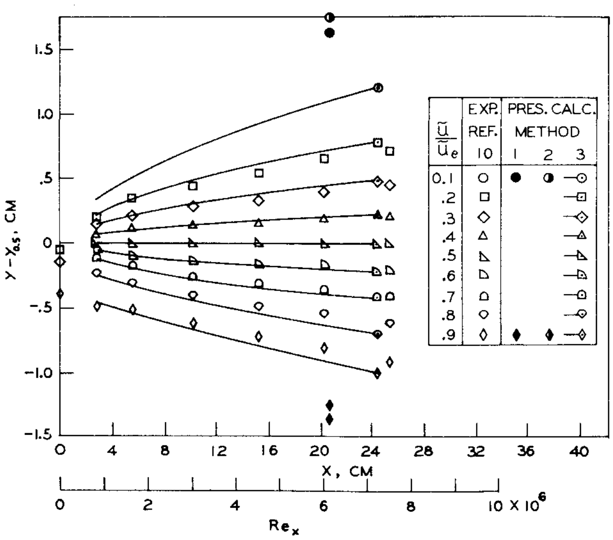

Fig. 2 Mean axial velocity, predictions, and measurements.

$$\bar{\rho}\tilde{u}\,\partial\tilde{H}/\partial x+\bar{\rho}\tilde{v}\,\partial\tilde{H}/\partial y+y^{-j}\partial y^{j}(\overline{\rho H'v'}-Pr^{-1}R_e^{-1}\bar{\mu}\,\partial\tilde{T}/\partial y-R_e^{-1}\bar{\mu}\,\tilde{u}\,\partial\tilde{u}/\partial y)/\partial y=0 \quad (4)$$

where $Pr = c_p\bar{\mu}/\bar{k}$, $R_e = \bar{\rho}_e\tilde{u}_e L/\bar{\mu}_e$, $j = 0$ for two-dimensional flow, $j = 1$ for axisymmetric flow, and subscript e refers to conditions on the high-velocity side of the shear layer. In the above it is assumed that, without change of notation, all variables are made dimensionless as follows: all lengths are referred to L, velocities to $\tilde{u}_e$, pressure to $\bar{\rho}_e\tilde{u}_e^2$, density to $\bar{\rho}_e$, temperature to $\tilde{u}_e^2/c_p$, total enthalpy to $\tilde{u}_e^2$, and viscosity to the value of μ at $T = T_e$.

In the previous equations, mass weighted temporal mean values[6] for velocity, total enthalpy, and temperature are denoted by tilde, and conventional time average values for density and molecular transport coefficients are represented by an over-bar.

Equations (1–4) and the equation of state

$$\bar{p} = \bar{\rho}\tilde{T}(\gamma-1)/\gamma \quad (5)$$

describe free turbulent shear flows completely with proper closure assumptions for the turbulent correlation terms.

The molecular transport terms, which are usually negligible in fully turbulent flow, are retained so that laminar flow problems may be computed without any added difficulty in numerical solution procedures.

The closure assumptions are made on turbulence structure and mixing length l as follows:

$$\overline{\rho u'v'} = \bar{\rho}l^2|\partial\tilde{u}/\partial y|\partial\tilde{u}/\partial y \quad (6)$$

$$\overline{\rho v'^2} = c|\overline{\rho u'v'}| \quad (7)$$

$$\overline{\rho H'v'} = \bar{\rho}\overline{T'v'}+\bar{\rho}\tilde{u}\overline{u'v'} \quad (8)$$

$$Pr_t \equiv (\overline{u'v'}/\partial\tilde{u}/\partial y)/(\overline{T'v'}/\partial\tilde{T}/\partial y) \quad (9)$$

$$l = l_m F \quad (10)$$

$$F = 1+\alpha(\partial\tilde{u}_i/\partial x_i)/|\partial\tilde{u}/\partial y| \quad (11)$$

For fully developed turbulence c and Pr_t are assumed constant with values of 1.3 and 0.9, respectively.[7,8]

F is a correction factor derived from Bradshaw's work[5] for the mean dilatation effect on the turbulence production term in the shear stress transport equation. Here we will use F with $\alpha = -8$ (Refs. 5 and 9) to modify the conventional mixing length l_m as in Eq. (10) (this is the correct interpretation if flow is locally in equilibrium). Rudy and Bushnell gave the following values for mixing length

$$l_m/\delta = 0.07 \quad (12)$$

for the two-dimensional shear layer where $\delta = 1.425 b_{0.05}$. The quantity $b_{0.05}$ is the width of the mixing zone measured between stations where $\tilde{u}/\tilde{u}_e = 0.05$ and $\tilde{u}/\tilde{u}_e = 0.95$.

Equations (1–5) together with closure assumptions (6–12) are solved by an implicit (Crank-Nicholson) finite-difference method. Nondimensional variables are used in their primitive form in the solution procedure. The normal momentum equation was coupled through an iteration procedure and the finite-difference expressions were linearized.

Equations (6, 4, 5, 2, and 1) are solved sequentially for $\overline{\rho u'v'}$, $\tilde{H}$, $\bar{p}$, $\tilde{u}$ and $\tilde{v}$, respectively. The latest updated variables are used in each of the equations. This procedure is repeated until $\tilde{u}$ and $\tilde{v}$ converge. Then the calculation is advanced to the next x step and $\bar{p}$ is computed from Eqs. (3) and (7) before the sequential procedure described above begins. New F values were computed once every two steps at the end of the iterative cycle.

Results and Discussion

In order to evaluate the method, calculations are compared with Morrisette's[10] Mach 5 turbulent freejet measurements. Computations were started at $x = 2.54$ cm with measured profiles as initial conditions. Initial profiles of $\bar{P}$ were generated from $\tilde{u}$ profiles since measured values were not available (measured $\bar{u}$ values were used as $\tilde{u}$). Constant values the same as initial conditions at both edges were used as boundary conditions.

Three different methods were used in the calculations to assess the effect of transverse static pressure variation and mean-dilatation corrections separately. These methods are: 1) mixing length only without normal momentum equation, 2) mixing length and normal momentum equation, and 3) mixing length

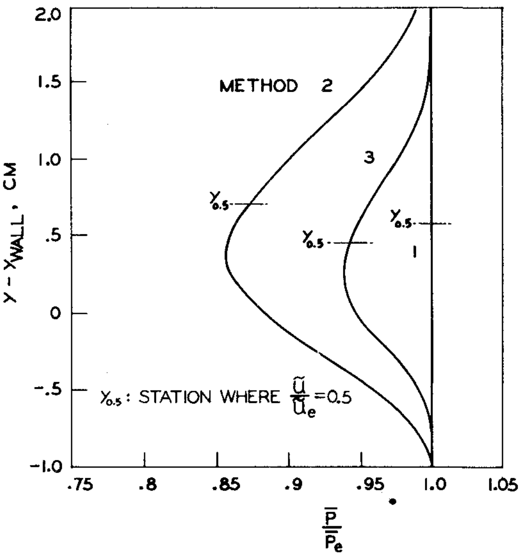

Fig. 3 Static pressure at $x = 10$ cm.

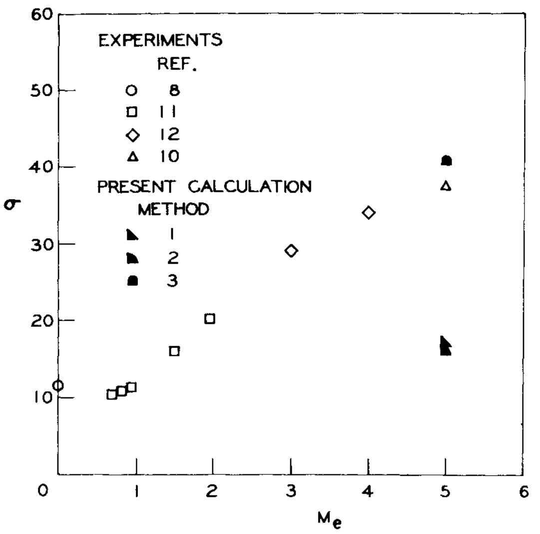

Fig. 4 Spreading parameter.

with a mean-dilatation correction factor and the normal momentum equation. In Method (3), an average value at a given x station, defined as

$$\bar{F}(x) = \int_{y_1}^{y_1} F\, dy/(y_2 - y_1)$$

has been used to avoid numerical difficulties. A typical computed variation of F is shown in Fig. 1 and the corresponding $\bar{F}$ which was used in the calculations. This average $\bar{F}$ was taken only over the portion of the shear layer where momentum diffusion is important as indicated by region between y_1 and y_2 in Fig. 1. Outside of this region the small values of momentum or velocity gradient caused the diffusivity correction to be of no significance in the solution. The value of $\bar{F}$ was found to vary from 0.703 at $x = 7.3$ cm to 0.553 at $x = 24.4$ cm.

Predicted mean axial velocity, static pressure, and spreading parameter σ are shown in Figs. 2–4 respectively.[11,12] σ has been defined by the relation[4] $\sigma = 1.32/\Delta\eta$ where $\Delta\eta$ is the angular distance between two rays η_1 and η_2 defined by $[\tilde{u}(\eta_1) - \tilde{u}_\infty]/(\tilde{u}_e - \tilde{u}_\infty) = (0.1)^{1/2}$ and $[\tilde{u}(\eta_2) - \tilde{u}_\infty]/(\tilde{u}_e - \tilde{u}_\infty) = (0.9)^{1/2}$. The inclusion of the normal mean momentum equation alone, without considering effects of pressure variation on the turbulence structure itself, does not improve the prediction. But simple mixing length theory with a mean dilatation effect correction, though it lacks sound analytical backing, predicts the turbulent free shear layer surprisingly well as can be seen in Figs. 2 and 4. It should be noted that the inclusion of the mean dilatation "correction factor" is in some sense a crude attempt to include the influence of the important $\overline{p' \partial u_i'/\partial x_i}$ term which appears in the compressible form of the turbulent kinetic energy equation. This term is expected to reduce $\overline{u'v'}$ and one would expect that generally $\partial u_i'/\partial x_i$ would be proportional to $\partial \tilde{u}_i/\partial x_i$.

References

[1] Rudy, D. H. and Bushnell, D. M., "A Rational Approach to the Use of Prandtl's Mixing Length Model in Free Turbulent Shear Flow Calculations," *Proceedings of the Conference on Free Turbulent Shear Flows*, NASA SP-321, Vol. 1, 1972, pp. 67–137.

[2] Bushnell, D. M. and Beckwith, I. E., "Calculation of Nonequilibrium Hypersonic Turbulent Boundary Layers and Comparisons With Experimental Data," *AIAA Journal*, Vol. 8, No. 8, Aug. 1970, pp. 1462–1469.

[3] Beckwith, I. E. and Bushnell, D. M., "Calculation by a Finite-Difference Method of Supersonic Turbulent Boundary Layers With Tangential Slot Injection," TN D-6221, April 1971, NASA.

[4] Brown, G. and Roshko, A., "The Effect of Density Difference on the Turbulent Mixing Layer," AGARD CP-93, 1971, pp. 23-1 to 23-12.

[5] Bradshaw, P., "Anomalous Effects of Pressure Gradient on Supersonic Turbulent Boundary Layers," Aero Rept. 72-21, Nov. 1972, Imperial College of Science and Technology, London, England.

[6] Favre, A., "Statistical Equations of Turbulent Gases," *Problems of Hydrodynamics and Continuum Mechanics*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1969, pp. 231–266.

[7] Bradshaw, P., "The Turbulence Structure of Equilibrium Boundary Layers," *Journal of Fluid Mechanics*, Vol. 29, Pt. 4, 1967, pp. 625–645.

[8] Liepmann, H. W. and Laufer, J., "Investigations of Free Turbulent Mixing," TN-1257, Aug. 1947, NACA.

[9] Wilcox, D. C. and Alber, I. E., "A Turbulence Model for High Speed Flows," *Proceedings of 23rd Heat Transfer and Fluid Mechanics Institute*, San Fernando Valley State College, North Ridge, Calif., 1972, pp. 231–252.

[10] Morrisette, E. L. and Birch, S. F., "Mean Flow and Turbulence Measurements in a Mach 5 Shear Layer. Part I—The Development and Spreading of the Mean Flow," *Fluid Mechanics of Mixing, ASME*, 1973, pp. 79–81.

[11] Maydew, R. C. and Reed, J. F., "Turbulent Mixing of Axisymmetric Compressible Jet (in the Half Jet Region) With Quiescent Air," Res. Rept. SC-4764 (RR), March 1963, Sandia Corp., Albuquerque, N.Mex.

[12] Sirieix, M. and Solignac, J. L., "Contribution à l'Etude Expérimentale de la Couche de Mélange Turbulent Isobare d'un Encoulement Supersonique," *Symposium on Separated Flow, AGARD* Conference Proceedings No. 4, 1966, pp. 241–270.

Effect of Finite Chemical Reaction Rates on Heat Transfer to the Walls of Combustion-Driven Supersonic MHD Generator Channels

J. W. DAILY*
Stanford University, Stanford, Calif.

AND

J. RAEDER† AND G. ZANKL†
Max-Planck-Institut für Plasmaphysik, Garching, Germany

THE effect of finite-rate homogeneous chemical reactions on heat-transfer rates to rocket nozzles is well known. It has not been widely recognized, however, that these effects may be important in combustion-driven supersonic MHD power generators, where conditions are similar to those in rocket nozzles. Data taken at the Institut für Plasmaphysik in Garching, Germany, and preliminary boundary-layer calculations done at Stanford Univ. indicate a significant reduction in wall heat flux due to finite rate effects.

Finite reaction rates may manifest themselves in two ways. First, the expansion process may be rapid enough to cause freezing in the bulk of the flow; secondly, gradients in the wall boundary layer may be severe enough, especially in turbulent flows, to cause freezing in the wall region. Comparison of reaction times[1] and the various flow characteristic times indicates that it is the latter case which is likely to occur for the experiments considered here.

Received August 24, 1973. The computer work reported was supported by AFAPL Contract F33615-72-C-1088.

Index categories: Boundary Layers and Convective Heat Transfer—Turbulent; Plasma Dynamics and MHD; Electric Power Generation Research.

* Research Assistant, High Temperature Gasdynamics Laboratory, Mechanical Engineering Department.

† Research Scientist.